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EPIMORPHISMS AND IDEALS OF 0- DISTRIBUTIVE SEMILATTICES

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ABSTARCT:

Preservation of the images and the inverse images of special types of ideals of a 0-distributive semilattice under an epimorphism with a condition on its kernel is established.

Mathematics Subject Classification: 06B99.

Keywords: 0- distributive lattice, ideal, homomorphism, α -ideal, 0-ideal, annihilator ideal.

INTRODUCTION:

As ageneralization of distributive lattices and pseudo-complemented lattices, 0- distributive lattices are introduced by Varlet [5].0-distributive semilattices arise as a natural generalisation of 0- distributive lattices. 0- ideals, annihilator ideals and α- ideals are special ideals introduced and studied in 0-distributive lattices by many researchers (see [1],[2],[3]and [4]).Analogously we have these special ideals in 0-distributive semilattices. It is well known that homomorphisms and their kernels play an important role in abstract algebra. In this paper our aim is to discuss about preservation of the images and the inverse images of these special ideals of a 0-distributive semilattice under an epimorphism with a condition that its kernel contains the smallest element only.

2. Preliminaries

Following are some basic concepts needed in sequel. By a semilattice S we mean meet semilattice$\langle S, \wedge\rangle$. The smallest (largest) element in S if exists is denoted by 0 (1).If both 0 and 1 exist in S, we say S is bounded. The semilattice S is said to be directed above, if for any a, $b \in S$, there exists $c\in L$ such that $c \geq a, b$.The semilattice$S$ with 0 is called 0- distributive, if for all$a$, $b\in S$, $a\wedge$ b = 0 = $a\wedge c$, then there exists $d \in S$such that $d \geq b$, c and $a\wedge d = 0$. Obviously, every 0- distributive semilattice is directed above. A non-empty subset A of S is a sub semilattice of S if $a\wedge b\in A$ for all $a,b\in A$. In a directed above semilattice S we have

(i) anon-empty subset A of S is a down set of S, if for $a, b \in S$, $a \leq b$, $b \in A$ imply $a \in A$.

(ii) a down set I of S is an ideal if for a, $b \in I$, there exists $c\in I$ such that $c \geq a$, b.

(iii) For $a\in S$, $(a] = \{ x \in S : x \leq a\}$denotes the smallest ideal in S containing $\{a\}$.

(iv) A non-empty subset F of S is a up set of S, if for a, $b \in S$, $a \leq b$, $a\in F$ imply $b\in F$.

(v) An up set F of S is a filter of S, if $a \wedge b \in F$ for all a, $b \in F$.

Notation: Throughout this paper S and S' will denote bounded 0-distributive semilattices with bounds {0, 1} and {0′, 1′ } respectively. By a homomorphism (i.e. a semilattice homomorphism) we mean a mapping $f: S \to S'$ satisfying: (i) $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in S$ and (ii) $f(0) = 0'$ and $f(1) = 1'$. The kernel of f is the set $\{x \in S : f(x) = 0'\}$ and we denote it by *Ker f*. Now we state lemma without proof which we use freely in the following sections.

Lemma2.1: Let S and S' be bounded 0- distributive semilattices and let $f: S \to S'$ be an epimorphism. Then we have

(i) For any filter F of S, $f(F)$ is a filter of S'.

(ii) For any ideal I of S, $f(I)$ is an ideal of S'.

(iii) For any filter F' of S', $f^{-1}(F')$ is a filter of S.

(iv) For any ideal I' of S', $f^{-1}(I')$ is a ideal of S.

(v) $Ker f$ is an ideal in S.

Remark 2.2: If f is a ring epimorphism, then it is an isomorphism if and only if kernel of f is $\{0\}$. But it is not true in the case of distributive lattices and hence for 0-distributive semilattices also. It may be seen from the following example.

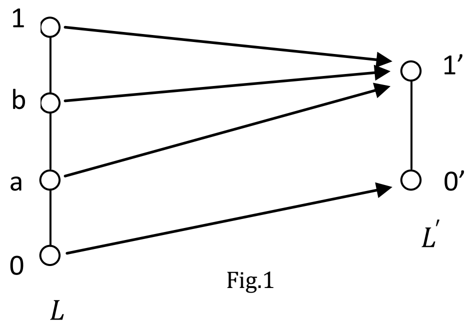


Fig.1

Consider the bounded distributive lattices $L = \{0, a, b, 1\}$ and $L' = \{0', 1'\}$ as shown by the Hasse Diagram of Fig.1. Define $f: L \to L'$ as shown in the figure. Clearly, f is onto and $Ker\ f = \{0\}$, but f is not one–one.

3.Epimorphisms and0- Ideals

We begin with the following definitions.

Definition 3.1: For any filter F of a semilattice S with 0, define $0(F) = \{x \in S : x \wedge f = 0, \text{for some } f \in F\} = \bigcup_{f \in F} \{f\}^*$.

Remarks:(i) $0(F)$ is an ideal in S and (ii) $0([x)) = (x]^* = \{x\}^*$.

Definition 3.2: An ideal I in S is called a 0-ideal if $I = 0(F)$, for some filter F of S.

Theorem 3.3: Let $f: S \to S'$ be an epimorphism. If $Ker\ f = \{0\}$, then we have

(i) $f(0(F)) = 0(f(F))$ for any filter F in S.

(ii) $0(F) = 0(G)$ if and only if $0(f(F)) = 0(f(G))$, for any filters F and G in S.

(iii) $f^{-1}(0(F')) = 0(f^{-1}(F'))$ for any filter F' in S'.

Proof: (i) Let F be any filter in S. Then $f(F)$ is a filter in S' (by Lemma 2.1). Take $x \in 0(F)$. Then $x \wedge y = 0$, for some $y \in F$. Now $x \wedge y = 0 \Rightarrow f(x \wedge y) = f(0) \Rightarrow f(x) \wedge f(y) = 0' \Rightarrow f(x) \in 0(f(F))$ (since $f(y) \in f(F)$). Hence $f(0(F)) \subseteq 0(f(F))$. Conversely, let $f(x) \in 0(f(F))$ for some $x \in S$ (as $S' = f(S)$). Then $f(x) \in 0(f(F)) \Rightarrow f(x) \wedge f(y) = 0'$ for some $y \in F \Rightarrow f(x \wedge y) = 0'$ for some $y \in F \Rightarrow x \wedge y \in Ker\ f = \{0\} \Rightarrow x \wedge y = 0 \Rightarrow x \in 0(F)$. This shows that $0(f(F)) \subseteq f(0(F))$. From both the inclusions we get $f(0(F)) = 0(f(F))$.

(ii) Let F and G be any filters in S such that $0(F) = 0(G)$. Then $0(F) = 0(G) \Rightarrow f(0(F)) = f(0(G)) \Rightarrow 0(f(F)) = 0(f(G))$ (by (i)). For the reverse implication assume that $0(f(F)) = 0(f(G))$ for the filters F and G in S. Then $x \in 0(F) \Rightarrow f(x) \in f(0(F)) \Rightarrow f(x) \in 0(f(F))$ (by (i)) $\Rightarrow f(x) \in 0(f(G)) \Rightarrow f(x) \wedge f(y) = 0'$ for some $y \in G \Rightarrow f(x \wedge y) = 0'$ for some $y \in G \Rightarrow x \wedge y \in Ker\ f = \{0\} \Rightarrow x \wedge y = 0 \Rightarrow x \in 0(G)$. This shows that $0(F) \subseteq 0(G)$. Similarly, we can prove that $0(G) \subseteq 0(F)$. Hence $0(F) = 0(G)$.

(iii) Let F' be a filter in S'. Let $x \in f^{-1}(0(F'))$. Then $f(x) \in 0(F') \Rightarrow f(x) \wedge f(y) = 0'$ for some $f(y) \in F' \Rightarrow f(x \wedge y) = 0'$ for some $y \in f^{-1}(F') \Rightarrow x \wedge y \in Ker\ f = \{0\} \Rightarrow x \wedge y = 0 \Rightarrow x \in 0(f^{-1}(F'))$. This shows that $f^{-1}(0(F')) \subseteq 0(f^{-1}(F'))$. Conversely, let $x \in 0(f^{-1}(F'))$. Then $x \wedge y = 0$, for some $y \in f^{-1}(F') \Rightarrow f(x \wedge y) = 0'$ for some $y \in f^{-1}(F') \Rightarrow f(x) \wedge f(y) = 0'$ for some $f(y) \in F' \Rightarrow f(x) \in 0(F') \Rightarrow x \in f^{-1}(0(F'))$. This shows that $0(f^{-1}(F')) \subseteq f^{-1}(0(F'))$. Combining both the inclusions we get $f^{-1}(0(F')) = 0(f^{-1}(F'))$. ■

In the following theorem we prove that the images and the inverse images of 0- ideals of a 0-distributive semilattice under an epimorphism with $Ker\ f = \{0\}$ are 0-ideals.

Theorem 3.4: Let $f: S \to S'$ be an epimorphism. If $Ker f = \{0\}$, then we have

(i)If K is a 0- ideal of S, then $f(K)$ is a 0-ideal of S'.

(ii) If K' is a 0- ideal of S', then $f^{-1}(K')$ is a 0-ideal of S.

Proof: (i) Let K be a 0-ideal of S. Then $K = 0(F)$, for some filter F in S. Hence $f(K)= f(0(F)) = 0\,(f(F))$ (by Theorem 3.3(i)).As $f(F)$ is a filter in S' (see Lemma 2.1),$f(K)$is a 0-ideal of S.

(ii) Let K' be a 0- ideal of S'. Then $K'= 0\,(F')$, for some filter F' in S. Hence $f^{-1}(K') = f^{-1}(0(F')) = 0(f^{-1}(F'))$ (by Theorem 3.3(iii)).As$f^{-1}(F')$ is a filter in S (see Lemma 2.1), $f^{-1}(K')$ is a 0-ideal of S. ■

4.Epimorphisms and Annihilator Ideals

We begin with the following definitions.

Definition 4.1: For any non-empty subset A of S, define

$$A^* = \{x \in S : x \wedge a = 0, \text{for each } a \in A\}.$$

A^* is called annihilator of A.

Remarks:

(i) If $A = \{a\}$, then $\{a\}^* = (\,a\,]^*$.

(ii) A directed above semilattice with 0 is 0-distributive if and only if A^* is an ideal in S, for any non-empty sub set A of S.

Definition 4.2: An ideal I of S is called an annihilator ideal if $I = A^*$, for some non-empty subset A of S or equivalently, $I = I^{**}$.

Theorem 4.3: Let $f: S\to S'$ be an epimorphism. If $Ker f = \{0\}$, then we have

(i) $f(A^*) = (f(A))^*$, for any non-empty subset $A$of$S$.

(ii) $f^{-1}(B^*) = (f^{-1}(B))^*$, for any non-empty subset BofS'.

(iii) $A^* = B^*$ if and only if $(f(A))^* = (f(B))^*$, for any non-empty subsets A and B of S. Proof: (i) Let A any non-empty subset of S. Let $f(x)\in f(A^*)$. Then $x \in A^* \Rightarrow x\wedge a = 0$ for each $a\in A \Rightarrow f(x\wedge a) = 0'$ for each $a\in A \Rightarrow f(x)\wedge f(a) = 0'$for each $f(a)\in f(A) \Rightarrow f(x)\in (f(A))^*$. Hence $f(A^*) \subseteq (f(A))^*$. Conversely, let $x' \in (f(A))^*$. Hence $x'= f(x)$, for some $x\in S$. As$f(x)\in (f(A))^* \Rightarrow f(x)\wedge f(a) = 0'$ for each $f(a)\in f(A) \Rightarrow f(x\wedge a) = 0'$ for each $a\in A \Rightarrow x\wedge a \in Ker f = \{0\}$for each $a\in A \Rightarrow x\wedge a = 0$ for each $a\in A \Rightarrow x\in A^* \Rightarrow f(x)\in f(A^*)$. This shows that $(f(A))^* \subseteq f(A^*)$. Combining both the inclusions,we get $f(A^*) = (f(A))^*$.

(ii) Let B beany non-empty subset ofS'. Let $x\in f^{-1}(B^*)$. Then $f(x)\in B^* \Rightarrow f(x)\wedge f(b) = 0'$for each $f(b)\in B \Rightarrow f(x\wedge b) = 0'$ for each $b\in f^{-1}(B) \Rightarrow x\wedge b \in Ker f = \{0\}$for each $b\in f^{-1}(B) \Rightarrow x\wedge b = 0$ for each $b\in f^{-1}(B) \Rightarrow x \in (f^{-1}(B))^*$. Hence $f^{-1}(B^*) \subseteq (f^{-1}(B))^*$.Conversely, let$x \in (f^{-1}(B))^*$.Then$x\wedge b = 0$ for each $b\in f^{-1}(B) \Rightarrow f(x\wedge b) = 0'$ for each $b\in f^{-1}(B) \Rightarrow f(x)\wedge f(b) = 0'$for each$f(b)\in B \Rightarrow f(x)\in B^* \Rightarrow x \in f^{-1}(B^*)$. Hence $(f^{-1}(B))^* \subseteq f^{-1}(B^*)$. Combining both the inclusions ,we get$f^{-1}(B^*) = (f^{-1}(B))^*$.

(iii) Let A and B be any non-empty subsets of S. Then $A^* = B^* \Rightarrow f(A^*) = f(B^*) \Rightarrow (f(A))^* = (f(B))^*$ (by (i)). Let$(f(A))^* = (f(B))^*$.Now $x \in A^* \Rightarrow x\wedge a = 0$ for each$a\in A \Rightarrow f(x\wedge a) = 0'$ for each $a\in A \Rightarrow f(x)\wedge f(a)= 0'$ for each $f(a)\in f(A) \Rightarrow f(x)\in (f(A))^* \Rightarrow f(x)\in (f(B))^* \Rightarrow f(x)\wedge f(b)= 0'$for each $f(b)\in f(B) \Rightarrow f(x\wedge b) = 0'$ for each $b\in B \Rightarrow x\wedge b \in Ker f = \{0\}$for each $b\in B \Rightarrow x\wedge b = 0$ for each $b\in B \Rightarrow x \in B^*$. This shows that $A^* \subseteq B^*$. Similarly, we can show that $B^* \subseteq A^*$.From both the inclusions we get $A^* = B^*$. ■

In the following theorem we prove that the images and the inverse images of annihilator ideals of a 0-distributive semilattice under an epimorphism with $Ker f = \{0\}$ are annihilator ideals.

Theorem 4.4: Let $f: S\to S'$be an epimorphism. If $Ker f = \{0\}$, then we have

(i) For any annihilator ideal I of S, $f(I)$ is an annihilator ideal of S'.

(ii)For any annihilator ideal $I'$of S', $f^{-1}(I')$ is an annihilator ideal of S.

Proof:(i) Let Ibe an annihilator ideal of S. Then $f(I)$ is an ideal in S'(see Lemma 2.1). Further$I = I^{**} \Rightarrow f(I) = f(I^{**}) \Rightarrow f(I) = \{f(I)\}^{**}$ (By Theorem 4.3 (i)). This shows that$f(I)$is an annihilator ideal of S'.

(ii) Let I' be an annihilator ideal of S'. Then $f^{-1}(I')$ is an ideal in S (see Lemma2,1). Further$I' = I'^{**} \Rightarrow f^{-1}(I') = f^{-1}(I'^{**}) \Rightarrow f^{-1}(I') = \{f^{-1}(I')\}^{**}$(ByTheorem 4.3 (ii)). This shows that $f^{-1}(I')$is an annihilator ideal of S.■

5.Epimorphismsand α-ideals

We begin with the following definition.

Definition 5.1: An ideal I in S is an α-ideal if $\{a\}^{**} \subseteq I$ for each $a \in I$.

Remark 5.2: Every annihilator ideal in S is an α-ideal.

We need the following characterizationof an α- ideal in a 0- distributive semilattice for proving our main results of this section.

Theorem5.3: Following statements are equivalent for an ideal$I$ in S.

(i) I is an α-ideal.

(ii) For $x, y \in S, \{x\}^* = \{y\}^*, x \in I \Rightarrow y \in I$.

Proof: (i) ⇒(ii) Suppose there exist $x, y \in S, \{x\}^* = \{y\}^*$, $x \in I$ but $y \notin I$. As I is an α-ideal in S, $\{x\}^{**} \subseteq I$.Therefore $y \notin \{x\}^{**}$ and hence $t \wedge y \neq 0$ for some $t \in \{x\}^*$. As $\{x\}^* = \{y\}^*$, we get $t \wedge y = 0$:which is not true. Hence $y \in I$ and the implication follows.

(ii) ⇒ (i) Let $x \in I$ and $t \in \{x\}^{**}$. Thus $t \wedge x \in I$ (since I is a down set) and $\{t\}^* \subseteq \{t \wedge x\}^*$. Again $y \in \{t \wedge x\}^* \Rightarrow t \wedge x \wedge y = 0 \Rightarrow t \wedge y \in \{x\}^* \Rightarrow t \wedge y \in \{t\}^*$ (since $t \in \{x\}^{**}$ implies $\{x\}^* \subseteq \{t\}^*$) . This gives $t \wedge y = 0$ and hence $y \in \{t\}^*$. Thus $\{t \wedge x\}^* \subseteq \{t\}^*$. Combining both the inclusions we get $\{t \wedge x\}^* = \{t\}^*$. As $t \wedge x \in I$, by assumption, $t \in I$. This shows that $\{x\}^{**} \subseteq I$ and hence I is an α-ideal in S.This completes the proof. ■

In the following theorem we prove that the images and the inverse images of α- ideals of a 0-distributive semilattice under an epimorphism with $Ker\ f = \{0\}$ are α-ideals.

Theorem 5.4: Let $f: S \to S'$ be an epimorphism. If $Ker\ f = \{0\}$, then we have

(i) If I is an α- ideal in S, then $f(I)$ is an α- ideal in S'.

(ii) If I' is an α- ideal in S', then $f^{-1}(I')$ is an α- ideal in S.

Proof: (i) Let I be an α-ideal in S. Then $f(I)$ is an ideal in S' by Lemma 2.1(i). Let $x \in f(I)$. Then $x = f(a)$ for some $a \in I$. As I is an α-ideal in S, $\{a\}^{**} \subseteq I$. Hence $f(\{a\}^{**}) \subseteq f(I) \Rightarrow (\{f(a)\}^{**}) \subseteq f(I)$ (by Theorem 4.3 (i)) $\Rightarrow \{x\}^{**} \subseteq f(I)$. Hence $f(I)$ is an α- ideal in S'.

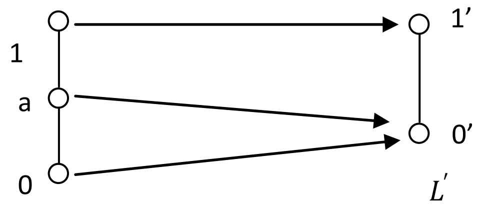
(ii) Let I' be an ideal in S'. Then $f^{-1}(I')$ is an ideal in S by Lemma 2.1(ii). Let x, y ∈ S such that $\{x\}^* = \{y\}^*$ and $x \in f^{-1}(I')$. But then $\{x\}^* = \{y\}^* \Rightarrow \{f(x)\}^* = \{f(y)\}^*$ (By Theorem 4.3 (iii)). As $f(x) \in I'$ and I' is an α-ideal in S', $f(y) \in I'$ i.e. $y \in f^{-1}(I')$.Hence $f^{-1}(I')$ is an α-ideal in S(by Theorem 5.3). ■

Theorem 5.5: Let $f: S \to S'$ is an epimorphism. Then for an α- ideal I' in S', $f^{-1}(I')$ is an α- ideal in S provided $f^{-1}(\{x'\}^*)$ is an α- ideal in S for any x' in S'.

Proof: Let I' be an α- ideal in S'. Then $f^{-1}(I')$ is an ideal in S (by Lemma 2.1(ii)). Let $x, y \in S$ such that $\{x\}^* = \{y\}^*$ and $x \in f^{-1}(I')$.Let $f(t) \in \{f(x)\}^*$ for some $t \in S$. Hence $f(x) \in \{f(t)\}^* \Rightarrow x \in f^{-1}(\{f(t)\}^*)$. By assumption, $f^{-1}(\{f(t)\}^*)$ is an α-ideal in S. Thus $\{x\}^* = \{y\}^*$ and $x \in f^{-1}(\{f(t)\}^*)$ imply $y \in f^{-1}(\{f(t)\}^*)$ (by Theorem 5.3). Thus $f(t) \wedge f(y) = 0' \Rightarrow f(t) \in \{f(y)\}^*$.This shows that $\{f(x)\}^* \subseteq \{f(y)\}^*$. Similarly we can show $\{f(y)\}^* \subseteq \{f(x)\}^*$. Hence $\{f(x)\}^* = \{f(y)\}^*$.As $f(x) \in I'$ and I' is an α- ideal in S', $f(y) \in I'$(by Theorem 5.3).Thus $y \in f^{-1}(I')$) and the result follows by Theorem 5.3) ■

Remark: The condition that $Ker\ f = \{0\}$ is indispensible in Theorems 3.4, 4.3 and 5.4. For this consider the following example.

Consider the bounded distributive lattices $L = \{0, a, 1\}$ and $L' = \{0', 1'\}$ as shown by the Hasse Diagram of Fig.2. Define $f: L \to L'$ as shown in the figure. Clearly, f is onto and $Ker\ f \neq \{0\}$,



L Fig. 2

For $= \{a, 1\}$, $f(0(F)) \neq$

For $A = \{a\}$, $f(A^*) \neq (f(A))^*$.

For α - ideal $\{0'\}$ in L', $f^{-1}(\{0'\})$ is not an α – ideal of L.

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